View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Linear Algebra and its Applications 433 (2010) 2122–2138

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Convex cones associated to generalized cones in $\mathbb{R}^N$ ☆

Luis A. Sanchez

Departamento de Matemática Aplicada y Estadística, Escuela Universitaria de Ingeniería Técnica Civil, Universidad Politécnica de Cartagena, 30203 Cartagena (Murcia), Spain

ARTICLE INFO

Article history:
Received 30 March 2009
Accepted 17 July 2010
Available online 21 August 2010

Submitted by H. Schneider

AMS classification:
15A48
15A75

Keywords:
Cones of rank k
Convex cones
Alternate multilinear forms

ABSTRACT

In this paper we associate to generalized cones of rank k in $\mathbb{R}^N$ certain convex cones in the space of alternate k-linear forms over $\mathbb{R}^N$. As a first consequence, we show a new proof of a Perron–Frobenius theorem for generalized positive operators. Afterwards, we give an application to the stability of periodic orbits of some monotone autonomous systems.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

Convex cones and positive operators have become undoubtedly one of the most extensively studied objects of linear algebra. The main reason for this is the enormous amount of applications that they find in pure and applied mathematics (see [2,7]). For instance the field of differential equations has employed the concepts of monotonicity and positivity to construct what is now known as the theory of monotone systems (see [6,12]). Many results concerning the dynamical behavior of these systems have been reported in recent years, exerting major impact especially in equations arising in biological models.

An outstanding property of positive operators is provided by the Perron–Frobenius theory. This shows the existence of dominant eigenvalues associated with positive eigenvectors and other relevant

☆ Supported by MTM2008-02502.
E-mail address: luis.sanchez@upct.es

0024-3795/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2010.07.004

spectral features. In [2,7] a wide account is given of the extreme importance of this property in applying positive operator theory to concrete problems.

Soon some researchers tried to extend the techniques related to positivity in order to study spectral properties of similar operators. A key advance is presented in [7] with the introduction of the new concept of cone of rank k (where k is a positive integer). A convenient extension of the notion of positive operator led to a generalized Perron–Frobenius theory that for example provided a new geometric insight into the classical work [5] by Gantmacher on totally positive operators.

In [10], we have taken a step forward in the use of these generalized cones through the introduction of a new class of monotone differential systems. More precisely, we have proven that monotone systems with respect to cones of rank 2 possess a Poincaré–Bendixson property. This allows us to extend several results on periodic orbits from the classical two-dimensional setting to any (finite) dimension.

Subsequent research has led us to establish a useful link between convex cones and generalized cones which have a strong incidence on the stability of periodic orbits. To be concrete, given C a cone of rank k in $\mathbb{R}^N$, we have constructed certain convex cones K in the space of k-linear alternate forms over $\mathbb{R}^N$ in such a way that positive operators with respect to C induce positive operators (in the usual sense) with respect to K. Our aim in this paper is to carry out this construction and indicate some of its applications. The first one is a new proof of the Perron–Frobenius results in [4,7]. This proof may offer some interest insofar as, unlike the preceding works, it has a pure geometric nature and does not rely on a fixed point argument.

We notice that k-linear alternate forms were employed in the above-mentioned work of Gantmacher and Krein (see also [1]). The reason is that total positivity has a straightforward formulation in terms of what they call skew-symmetric product of vectors. This paper can be viewed as a broad generalization of that theory through its combination with the geometrical perspective adopted in [7].

The paper is organized as follows. In next section, we summarize some properties of k-linear alternate forms. We mention some purely algebraic ones and offer a wider explanation of some geometrical and topological aspects that will be especially relevant in our work.

In the third section, we review the concept of cone of rank k ($k \in \mathbb{N}$), and we construct the associated convex cones in Λ_k the space of alternate k-linear forms. We also complete the basic definitions with some additional noteworthy properties of the cones.

Section 4 deals with the generalized positive operators and how they induce positive operators with respect to the associated convex cones in Λ_k. This will allow us to provide a new proof of a Perron–Frobenius theorem for such operators based on the classical Perron–Frobenius theory.

In Section 5, we apply our work to a very well-known example of a generalized cone defined by means of the number of sign changes in the coordinates of vectors in $\mathbb{R}^N$. Many of the results shown there are already present in [1,5,7], at least implicitly. Our aim is simply to reformulate them in order to fit them into our new general approach.

In the final section, we go back to our motivating problem, which is the study of the stability of periodic orbits of the new autonomous systems introduced in [10]. We merely indicate the applications we have in mind and leave further study to future works.

2. Some background on alternate multilinear forms

We first enumerate some basic properties of alternate multilinear forms. A thorough treatment of them can be found in [13].

Let Λ_k denote the space of alternate k-linear forms over $\mathbb{R}^N$. The decomposable elements of Λ_k are those written as $x^1 \wedge x^2 \wedge \cdots \wedge x^k$ where $x^i \in \mathbb{R}^N$. These forms act over vectors $y^1, \ldots, y^k \in \mathbb{R}^N$ according to the formula

$$x^1 \wedge \cdots \wedge x^k(y^1, \ldots, y^k) = \mathrm{Det}(\langle x^i, y^j \rangle).$$

($\langle \cdot, \cdot \rangle$ stands for the usual scalar product in $\mathbb{R}^N$).

Each element in Λ_k is a linear combination of decomposable k-forms. In fact, if $B = \{e^1, \ldots, e^N\}$ is a basis of $\mathbb{R}^N$, then

$$B_k = \{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant N\}$$

is a basis of Λ_k. In particular dim $\Lambda_k = \binom{N}{k}$.

The formula

$$\langle x^1 \wedge \cdots \wedge x^k, y^1 \wedge \cdots \wedge y^k \rangle = x^1 \wedge \cdots \wedge x^k(y^1, \ldots, y^k)$$

defines (through the obvious extension by linearity) a scalar product in Λ_k. If B is an orthogonal (resp. orthonormal) basis of $\mathbb{R}^N$, then B_k is an orthogonal (resp. orthonormal) basis of Λ_k.

To work with coordinates, let us introduce some notation concerning minors of matrices. Let $A = \left(a_i^j\right)$ be a matrix of order $N_1 \times N_2$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k)$ with positive integer components such that $1 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k \leqslant N_1$, $1 \leqslant \beta_1 < \beta_2 < \cdots < \beta_k \leqslant N_2$. Let m_α^β be the determinant of $\left(a_{\alpha_i}^{\beta_j}\right)$ with $i, j = 1, \ldots, k$. Giving the lexicographic ordering to the set of vectors with positive integer components, we can then construct the matrix $A^{(k)} = \left(m_\alpha^\beta\right)$, called the kth multiplicative compound of A.

Let $\left(x_1^i, \ldots, x_N^i\right)$ be the coordinates of x^i with respect to the basis B for $i = 1, \ldots, k$. Then the coordinates of $x^1 \wedge \cdots \wedge x^k$ with respect to the basis B_k of Λ_k is just the kth multiplicative compound of the matrix $M = (x_j^i)$.

Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a linear map. The kth multiplicative compound $T^{(k)} : \Lambda_k \to \Lambda_k$ is defined as

$$T^{(k)}(x^1 \wedge \cdots \wedge x^k) = Tx^1 \wedge \cdots \wedge Tx^k$$

over decomposable elements and extended linearly to all Λ_k. If A is the matrix representation of T with respect a basis B, then $A^{(k)}$ is the matrix representation of $T^{(k)}$ with respect to B_k.

Multiplicative compounds have two remarkable properties:

(a) Cauchy–Binet Formula: $(AB)^{(k)} = A^{(k)}B^{(k)}$.
(b) The spectrum of $A^{(k)}$ is

$$\mathrm{Sp}(A^{(k)}) = \{\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant N\},$$

where $\mathrm{Sp}(A) = \{\mu_1, \ldots, \mu_N\}$ is the spectrum of A.

The form $x^1 \wedge \cdots \wedge x^k$ is zero if and only if $x^1, \ldots, x^k$ are linearly dependent. Moreover the forms $x^1 \wedge \cdots \wedge x^k$ and $y^1 \wedge \cdots \wedge y^k$ (different from zero) are linearly dependent if and only if

$$L[x^1, \ldots, x^k] = L[y^1, \ldots, y^k],$$

where $L[u^1, \ldots, u^m]$ stands for the linear subspace generated by $\{u^1, \ldots, u^m\}$. As a consequence, every subspace of dimension k of $\mathbb{R}^N$ can be identified up to a nonzero constant with a decomposable element of Λ_k.

Now, in more detail, we discuss some properties of topological nature which will be useful later. First, the applications

$$(x^1, \ldots, x^k) \to x^1 \wedge \cdots \wedge x^k, \quad T \to T^{(k)}$$

are both continuous with respect to the standard topologies in each space.

In particular, given sequences $\left\{x_n^i\right\}$ converging to x_0^i for $i = 1, \ldots, k$, it holds that the sequence $\left\{x_n^1 \wedge \cdots \wedge x_n^k\right\}$ converges to $x_0^1 \wedge \cdots \wedge x_0^k$. We establish now a sort of reciprocal result in next proposition.

Proposition 1. *Let $\left\{x_n^1 \wedge \cdots \wedge x_n^k\right\}$ be a sequence of nonzero forms that converges to a certain $\lambda \in \Lambda_k - \{0\}$. Then there exist vectors $x_0^1, \ldots, x_0^k \in \mathbb{R}^N$ such that $\lambda = x_0^1 \wedge \cdots \wedge x_0^k$. In addition, for every $n \in \mathbb{N}$ there are vectors $w_n^1, \ldots, w_n^k \in L\left[x_n^1, \ldots, x_n^k\right]$ such that the sequence $\left\{w_n^i\right\}$ tends to x_0^i for $i = 1, \ldots, k$.*

Proof. For each $n \in \mathbb{N}$ we can apply the Gram–Schmidt orthogonalization process in $L\left[x_n^1, \ldots, x_n^k\right]$ and find vectors $y_n^i \in \mathbb{R}^N$ such that

$$x_n^1 \wedge \cdots \wedge x_n^k = y_n^1 \wedge \cdots \wedge y_n^k,$$
$$\left\langle y_n^i, y_n^j \right\rangle = 0 \quad \text{for } i \neq j$$

and

$$|y_n^1| = \cdots = |y_n^{k-1}| = 1.$$

The convergence to λ and the equality $|y_n^1 \wedge \cdots \wedge y_n^k| = |y_n^k|$ implies that $\{|y_n^k|\}$ must be bounded. Therefore the sequences $\left\{y_n^i\right\}$ have a common subsequence that converges to some x_0^i. Hence we deduce that

$$\lambda = x_0^1 \wedge \cdots \wedge x_0^k,$$

and the first part of the proposition is proven.

The second part would abide from the following property: for every $\epsilon > 0$ there exists n_0 such that for all $n > n_0$ the subspace $L\left[x_n^1, \ldots, x_n^k\right]$ intersects the balls

$$B\left(x_0^i, \epsilon\right) = \left\{x \in \mathbb{R}^N : \left|x - x_0^i\right| < \epsilon\right\}, \quad i = 1, \ldots, k.$$

We now deny this property and reach a contradiction, from which the proposition will hold true.

Thus we assume the existence of $\epsilon_0 > 0$, a superindex $i_0 \in \{1, \ldots, k\}$ and a subsequence $\left\{x_{n_j}^1 \wedge \cdots \wedge x_{n_j}^k\right\}$ verifying

$$L\left[x_{n_j}^1, \ldots, x_{n_j}^k\right] \cap B\left(y^{i_0}, \epsilon_0\right) = \emptyset. \tag{1}$$

As in the proof of the first part, we can construct subsequences $\{w_{n_m}^i\}$ such that

$$w_{n_m}^1 \wedge \cdots \wedge w_{n_m}^k = x_{n_m}^1 \wedge \cdots \wedge x_{n_m}^k$$

and each $\left\{w_{n_m}^i\right\}$ converging to a certain w_0^i. Obviously

$$w_0^1 \wedge \cdots \wedge w_0^k = x_0^1 \wedge \cdots \wedge x_0^k.$$

Thus there exist real numbers $\{\alpha_{ij}\}_{i=1,\ldots,k}^{j=1,\ldots,k}$ such that

$$x_0^i = \sum_{j=1}^{k} \alpha_i^j w_0^j, \quad i = 1, \ldots, k.$$

Let us consider the vector

$$z_{n_m}^{i_0} = \sum_{j=1}^{k} \alpha_{i_0}^j w_{n_m}^j \in L\left[x_{n_m}^1, \ldots, x_{n_m}^k\right].$$

Clearly $\left\{z_{n_m}^{i_0}\right\}$ tends to $\sum_{j=1}^k \alpha_{i_0}^j w_0^j = x_0^{i_0}$. This contradicts (1). $\square$

We notice that we can express the convergence of sequences of subspaces as follows: given $\{H_n\}$ a sequence of subspaces of dimension k, we say that it converges to the subspace H_0 if there are sequences $\left\{x_n^i\right\}$ such that $H_n = L\left[x_n^1, \ldots, x_n^k\right]$ verifying that

$$\left\{x_n^i\right\} \to x_0^i, \quad i = 1, \ldots, n$$

and

$$H_0 = L\left[x_0^1, \ldots, x_0^k\right].$$

Based on the above, we can assert that any such sequence $\{H_n\}$ has a converging subsequence.

3. Convex cones generated by cones of rank *k*

The following definition is taken from Krasnoselskij et al. [7].

Definition 1. A closed set $C \subset \mathbb{R}^N$ is a cone of rank k if

(1) It is homogeneous, i.e. $x \in C,\ \lambda \in \mathbb{R} \Rightarrow \lambda x \in C$.
(2) It contains a subspace of dimension k but no subspace of dimension greater than k.

The set $C^c = \overline{\mathbb{R}^n - C}$ is also a cone called the complementary cone of C.

We say that C is k-solid if there is a linear subspace Π of dimension k such that $\Pi - \{0\} \subset \overset{\circ}{C}$. We say that C is complemented if there is a linear subspace Π of dimension $N-k$ such that $\Pi^c \cap C = \{0\}$. In this case, C^c is a cone of rank $N-k$ that is $(N-k)$-solid.

In the sequel we fix a complemented cone C of rank k $(2 \leqslant k \leqslant N-1)$. Let us take Π and Π^c subspaces of dimension k and $N-k$ with

$$\Pi - \{0\} \subset \overset{\circ}{C} \quad \text{and} \quad \Pi^c - \{0\} \subset \mathbb{R}^N - C,$$

and take $B = \{e^1, \ldots, e^k\}$ and $B^c = \{e^{k+1}, \ldots, e^N\}$ bases of Π and Π^c, respectively. The set

$$B_1 = \{e^1, \ldots, e^k, e^{k+1}, \ldots, e^N\}$$

is a basis of $\mathbb{R}^N$. Let P be the hyperplane in Λ_k generated by the vectors $e^{i_1} \wedge \cdots \wedge e^{i_k}$ with $(i_1, \ldots, i_k) \neq (1, \ldots, k)$.

Let us define

$$C^{(k)} = \{u^1 \wedge \cdots \wedge u^k : L[u^1, \ldots, u^k] \subset C\}. \tag{2}$$

$C^{(k)}$ is closed in Λ_k since C is closed in $\mathbb{R}^N$. In addition, it is homogeneous, i.e.

$$\lambda \in C^{(k)}, \quad \alpha \in \mathbb{R} \Rightarrow \alpha\lambda \in C^{(k)}.$$

In particular it is symmetric in the sense that $\lambda \in C^{(k)} \Rightarrow -\lambda \in C^{(k)}$.

Lemma 1. *It holds that* $C^{(k)} \cap P = \{0\}$.

Proof. Let us take a form

$$u^1 \wedge \cdots \wedge u^k \in C^{(k)} \cap P.$$

Writing $u^i = \left(u_1^i, \ldots, u_N^k\right)$ the coordinates of u^i with respect to B, we know that

$$u^1 \wedge \cdots \wedge u^k = (u_j^i)^{(k)}$$

with respect to B_k. Since $u^1 \wedge \cdots \wedge u^k \in P$ we deduce that the minor $m_{1,\ldots,k}^{1,\ldots,k}$ is zero. From this, we can assert the existence of a nontrivial linear combination

$$v = c^1 u^1 + \cdots + c^k u^k$$

having the first k coordinates equal to 0. This means that

$$v \in \Pi^c = L\left[e^{k+1}, \ldots, e^N\right].$$

On the other hand, by construction of $C^{(k)}$ we know that $v \in C$. The hypothesis $C \cap \Pi^c = \{0\}$ yields to $v = 0$. Since v was a nontrivial linear combination of $u^1, \ldots, u^k$ we deduce that these vectors are linearly dependent and therefore $u^1 \wedge \cdots \wedge u^k = 0$ as desired. $\square$

Let P be any hyperplane of Λ_k (not necessarily constructed as before) such that $P \cap C^{(k)} = \{0\}$. To distinguish between the two hemispaces of Λ_k induced by P, we take F, a nontrivial functional over Λ_k, which is zero over P. We define the sets

$$C_+^{(k)}(P) = \{\lambda \in C^{(k)} : F(\lambda) > 0\}$$

and

$$C_-^{(k)}(P) = \{\lambda \in C^{(k)} : F(\lambda) < 0\}.$$

Trivially $C_+^{(k)}(P) = -C_-^{(k)}(P)$ and $C_+^{(k)}(P) \cup C_-^{(k)}(P) = C^{(k)} - \{0\}$. We define now

$$D = \{\lambda \in C_+^{(k)}(P) : F(\lambda) = 1\}.$$

D is closed since C is closed. We claim that D is also bounded. Otherwise, we would find a sequence $\{\lambda_n\}$ in $C_+^{(k)}(P)$ such that $F(\lambda_n) = 1$ and $\|\lambda_n\| \to +\infty$. By extracting a subsequence, we can assume that $\mu_n = \frac{\lambda_n}{\|\lambda_n\|}$ defines a sequence that converges to a unitary $\mu_0 \in \Lambda_k$. Again $\mu \in C_+^{(k)}(P)$ because C is closed. We now notice that $F(\mu_n) = 1/\|\lambda_n\|$ tends to zero, and thus $F(\mu_0) = 0$. This contradicts the fact that $C^{(k)} \cap P = \{0\}$.

Consequently, D is compact. Its convex envelop $\mathrm{co}(D) = \overline{\mathrm{co}}(D)$ is also compact. It is immediate that

$$\overline{\mathrm{co}}\left(C_+^{(k)}(P)\right) = \{\alpha\lambda : \alpha \geqslant 0, \lambda \in \mathrm{co}(D)\}.$$

Clearly

$$K(P) := \overline{\mathrm{co}}(C_+^{(k)}(P))$$

is closed, convex, homogeneous and satisfies $K(P) \cap (-K(P)) = \{0\}$. In short $K(P)$ is a convex cone.

Definition 2. The set $K(P)$ is called the convex cone in Λ_k associated to C induced by the hyperplane P.

Remark 1. Actually the same process applied to $C_-^{(k)}(P)$ yields another convex cone, just $-K(P)$. Thus, the preceding definition is somewhat ambiguous. Nevertheless, from here on, we obviate this point since it will not have any relevance in the statements and proofs that follow.

Remark 2. From the construction above, we deduce that if G is any nontrivial linear functional and $G(\lambda) > 0$ for all $\lambda \in C_+^{(k)}(P)$ then $G(\lambda) > 0$ for all $\lambda \in K(P)$.

Remark 3. If Q is a hyperplane close enough to P then $C^{(k)} \cap Q = \{0\}$. Furthermore $C_+^{(k)}(P) = C_+^{(k)}(Q)$ and thus $K(Q) = K(P)$.

In the sequel, we fix the convex cone $K(P)$, and denote it simply by K when P plays no role.

Lemma 2. *K has non-empty interior.*

Proof. If this was not so, and since K is convex, then K would be contained in an hyperplane of Λ_k. In fact $K \cup (-K)$ would be contained in such a hyperplane. It turns out that there exists F, a nontrivial linear form over Λ_k, such that F is zero over K. F induces an alternate k-linear form T through the formula

$$T(x^1, \ldots, x^k) = F(x^1 \wedge \cdots \wedge x^k).$$

Let us consider an element $y^1 \wedge \cdots \wedge y^k \in K$ such that

$$L[y^1, \ldots, y^k] - \{0\} \subset \overset{\circ}{C}.$$

It is straightforward that there exists $\epsilon > 0$ such that

$$L[x^1, \ldots, x^k] - \{0\} \subset \overset{\circ}{C} \quad (\text{whence } x^1 \wedge \cdots \wedge x^k \in K)$$

if $|x^i - y^i| < \epsilon$. Then we deduce that the k-linear form T is zero in a neighborhood of $(y^1, \ldots, y^k) \in \mathbb{R}^N \times \cdots \times \mathbb{R}^N$. This implies that T is actually zero, but this contradicts the fact that F is nontrivial. □

We can get further in next proposition.

Proposition 2. *A decomposable form $y^1 \wedge \cdots \wedge y^k \in K$ is in the interior of K provided that $L[y^1, \ldots, y^k] - \{0\} \subset \overset{\circ}{C}$.*

Proof. Otherwise $y^1 \wedge \cdots \wedge y^k$ would belong to the boundary of K. Thus, we could find a nonzero linear functional F such that $F(y^1 \wedge \cdots \wedge y^k) = 0$ and

$$F(x^1 \wedge \cdots \wedge x^k) \geqslant 0 \quad \text{for } x^1 \wedge \cdots \wedge x^k \in K.$$

We call T the nonzero multilinear map defined by F as in the preceding proof. Take $\epsilon > 0$ small enough such that

$$L[x^1, \ldots, x^k] - \{0\} \subset \overset{\circ}{C} \quad \text{if } |x^i - y^i| < \epsilon, \quad i = 1, \ldots, k.$$

It turns out that T has a local minimum at $(x^1, \ldots, x^k)$. Let us check that this is impossible.

First, we pick up an arbitrary vector $x \in \mathbb{R}^N$ and consider the function

$$\alpha(h) = T(x^1 + hx, x^2, \ldots, x^k), \quad h \in \mathbb{R}.$$

By hypothesis, α has a local minimum at $h = 0$. However,

$$\alpha(h) = T(x^1, \ldots, x^k) + hT(x, \ldots, x^k) = hT(x, \ldots, x^k).$$

Therefore, necessarily, $T(x, \ldots, x^k) = 0$. The same argument leads to the following property for T:

For any $j = 1, \ldots, k$, the linear form

$$T^k(x) = (x^1, \ldots, x^{j-1}, x, x^{j+1} \ldots, x^k)$$

is zero.

Let us now consider a couple of vectors $x, y \in \mathbb{R}^N$ and define the function

$$\beta(h_1, h_2) = T(x^1 + h_1 x, x^2 + h_2 y, x^3, \ldots, x^k), \quad h_1, h_2 \in \mathbb{R}.$$

Again, β has a local minimum at $(0, 0)$ by hypothesis. On the other hand,

$$\beta(h_1, h_2) = T(x^1, \ldots, x^k) + h_1 T(x, x^2, \ldots, x^k) + h_2 T(x^1, y, \ldots, x^k) + h_1 h_2 T(x, y, \ldots, x^k). \quad (3)$$

From what has been proved above, we deduce that

$$\beta(h_1, h_2) = h_1 h_2 T(x, y, \ldots, x^k).$$

Hence the only possibility for β to have a minimum at $h_1 = 0,\ h_2 = 0$ is that $T(x, y, x^3, \ldots, x^k) = 0$. Since $x,\ y$ are arbitrary, when we repeat the argument for every couple of components, we find that for every $i, j = 1, \ldots, k$ the bilinear form

$$B(x, y) = T(x^1, \ldots, x^{i-1}, x, x^{i+1}, \ldots, x^{j-1}, y, x^{j+1}, \ldots, x^k)$$

is zero. Immediately, by reiterating the argument, we find that T would be zero. However, this is a contradiction, since F was chosen to be nontrivial. □

The geometry of the cone C can be rather complex, unlike the case of convex cones. Let us introduce two properties of the associated convex cone K, which can be interpreted as a restriction of such a possible complex structure.

Definition 3. C is said to be full if $x^1 \wedge \cdots \wedge x^k \in K(P) - \{0\}$ implies that $L[x^1, \ldots, x^k] \subset C$.

Let $x^1 \wedge \cdots \wedge x^k \neq 0$ with $L[x^1, \ldots, x^k] \subset C$ but $L[x^1, \ldots, x^k] \cap \partial C \neq \{0\}$. We can take $y^1, \ldots, y^k$ such that

$$y^1 \wedge \cdots \wedge y^k = x^1 \wedge \cdots \wedge x^k \quad \text{and} \quad y^1 \in \partial C.$$

Let us consider a sequence $\left\{y_n^1\right\}$ converging to y^1 with $y_n^1 \in \mathbb{R}^N - C$. We find that

$$\left\{y_n^1 \wedge \cdots \wedge y^k\right\} \to y^1 \wedge \cdots \wedge y^k = x^1 \wedge \cdots \wedge x^k.$$

If C is full, then $y_n^1 \wedge \cdots \wedge y^k$ does not belong to K. Therefore $x^1 \wedge \cdots \wedge x^k$ belongs to the boundary of K. This fact, together with Proposition 2, leads to the following proposition.

Proposition 3. *If C is full then*

$$x^1 \wedge \cdots \wedge x^k \in \overset{\circ}{K} \Leftrightarrow L[x^1, \ldots, x^k] - \{0\} \subset \overset{\circ}{C}$$

and

$$x^1 \wedge \cdots \wedge x^k \in \partial K \Leftrightarrow L[x^1, \ldots, x^k] \subset C \quad \textit{and} \quad L[x^1, \ldots, x^k] \cap \partial C \neq \{0\}.$$

The next definition states a connectivity property that has strong implications for the uniqueness of these associated convex cones.

Definition 4. We say that C is k-connected if for some hyperplane P the set $C^{(k)} \cap K(P) - \{0\}$ is connected.

Lemma 3. *If C is k-connected, then $K(P)$ is independent of P.*

Proof. Let Q be another hyperplane of Λ_k with $Q \cap C^{(k)} = \{0\}$ and F a nontrivial linear functional which is zero over Q. By hypothesis $F(\lambda) \neq 0$ for all $\lambda \in C_+^{(k)}(P)$. If F is either positive or negative over $C_+^{(k)}$, then obviously $K(Q)$ is either equal to $K(P)$ or $-K(P)$. If F changes sign over $C_+^{(k)}(P)$ then Q separates $K(P)$, this contradicting the k-connectedness of $K(P)$. $\square$

4. Positive operators with respect to cones of rank k

We still consider a complemented cone C of rank k that is k-solid and continue to use the notation of the preceding section.

Proposition 4. *A linear operator $A : \mathbb{R}^N \to \mathbb{R}^N$ is said strongly positive with respect to C if $A(C - \{0\}) \subset \overset{\circ}{C}$.*

Lemma 4. *If A is inversible and strongly positive with respect to C, then A^{-1} is strongly positive with respect to C^c.*

Proof. First, we notice that, since $\mathbb{R}^N - C$ and $\overset{\circ}{C}$ are non-empty open sets, it holds that

$$\mathbb{R}^N - C \subset \overset{\circ}{C^c} \tag{4}$$

and

$$C^c \cap \overset{\circ}{C} = \emptyset. \tag{5}$$

Let us take $x \in C^c - \{0\}$ and let us assume that $A^{-1}x \notin \overset{\circ}{C^c}$. In particular, if we use (4), it should hold that $A^{-1}x \in C$. In fact, since $x \neq 0$ and A is inversible, we would have that $A^{-1}x \in C - \{0\}$. Since A is strongly positive with respect to C we find that $AA^{-1}x = x \in \overset{\circ}{C}$. Thus, $x \in (C^c - \{0\}) \cap \overset{\circ}{C}$, contradicting (5). $\square$

A key point in our work is the obvious inclusion

$$A^{(k)}C^{(k)} \subset C^{(k)}$$

provided that A is strongly positive. This appears to indicate that $A^{(k)}$ could be strongly positive with respect to $K(P)$. This would not be true so long as $A^{(k)}$ may mix the forms $x^1 \wedge \cdots \wedge x^k$ of the two hemispaces induced by the hyperplane P. Our next aim is to select one such hyperplane P for which these ideas can be carried out. First, we need to establish a lemma.

Lemma 5. *There exist two subspaces Π_0 and Π_0^c of dimension k and $N - k$ respectively satisfying*

$$A^n(\Pi_0) - \{0\} \subset \overset{\circ}{C}, \quad A^n(\Pi_0^c) - \{0\} \subset \mathbb{R}^n - C$$

for all integers n.

Proof. Let us take a subspace Π of dimension k with $\Pi \subset C$. Let us define the subspaces $\Pi_n = A^n(\Pi)$ for $n \in \mathbb{N}$. Since A is strongly positive, we find that $\Pi_n \subset A(C)$. Now let us observe that $A(C)$ is a cone of rank k, verifying

$$A(C) - \{0\} \subset \overset{\circ}{C}.$$

Consider a subsequence $\{\Pi_{n_j}\}$ converging to certain Π_0. Then

$$\Pi_0 - \{0\} \subset A(C) - \{0\} \subset \overset{\circ}{C}.$$

It is obvious that in fact

$$A^n(\Pi_0) - \{0\} \subset \overset{\circ}{C} \tag{6}$$

for all $n \in \mathbb{N}$. On the other hand, given $m \in \mathbb{N}$ arbitrary, we can assert that

$$A^{-m}(\Pi_0) = \lim_{n_j \to +\infty} A^{-m}(A^{n_j}(\Pi)) = \lim_{n \to +\infty} A^{n_j - m}(\Pi) \subset A(C),$$

where the last inclusion is true for $n_j > m$. Thus

$$A^{-m}(\Pi_0) - \{0\} \subset \overset{\circ}{C}.$$

This fact, together with (6), gives the desired property for Π_0.

The construction of Π^c is identical by reasoning with the strong positive operator A^{-1}. $\square$

Theorem 1. *Let us assume that A is a strongly positive operator with respect to C. Then there exist a cone of the form $K(P)$ and a natural number n_0 such that $G = A^{n_0}$ satisfies the condition that $G^{(k)}$ is strongly positive with respect to $K(P)$.*

Proof. We take the subspaces Π_0 and Π_0^c as in the preceding lemma and define

$$H_n = A^n(\Pi_0), \quad H_n^c = A^n(\Pi_0^c) \quad \text{for } n \in \mathbb{N}. \tag{7}$$

Let us consider a sequence of natural numbers $\{n_j\} \to +\infty$ as $j \to +\infty$ such that

$$\lim_{j\to+\infty} H_{n_j} = H_0 \quad \text{and} \quad \lim_{j\to+\infty} H^c_{n_j} = H^c_0.$$

It follows that

$$H_0 - \{0\} \subset \overset{\circ}{C} \quad \text{and} \quad H^c_0 - \{0\} \subset \mathbb{R}^N - C.$$

Let us consider the hyperplanes P_{n_j} of Λ^k induced by the couple H_{n_j}, $H^c_{n_j}$ and P_0 the one induced by H_0, H^c_0. Obviously, P_{n_j} tends to P_0. From this, we deduce a property which we describe in next paragraph.

We use P^+_0, P^-_0 to denote the closed hemispaces defined by P_0, and

$$C^+_0 = P^+_0 \cap C^{(k)}, \quad C^-_0 = P^-_0 \cap C^{(k)} = -C^+_0$$

(see the definitions in (2) and after Lemma 1). Similarly, we define the corresponding subsets $P^-_{n_j}$, $P^+_{n_j}$, $(C^+)_{n_j}$, $(C^-)_{n_j}$. The convergence of P_{n_j} to P_0 implies that for all sufficiently large n_j

$$(C^+)_{n_j} = C^+_0 \quad \text{or } (C^+)_{n_j} = C^-_0.$$

In particular, there are $n_p < n_q$ such that

$$(C^+)_{n_p} = (C^+)_{n_q} \quad \text{or } (C^+)_{n_p} = (C^-)_{n_q}.$$

Simply by changing names, we can assume that $(C^+)_{n_p} = (C^+)_{n_q}$.

Let us now take the operator $B = A^{n_q - n_p}$ and consider $B^{(k)}$. In the same manner as for the operator $A^{(k)}$ itself, we know that $B^{(k)}C^{(k)} \subset C^{(k)}$. On the other hand, by the definition of B and (7), we find that

$$BH_{n_p} = H_{n_q} \quad \text{and} \quad BH^c_{n_p} = H^c_{n_q}.$$

Hence the equality $B^{(k)}P_{n_p} = B^{(k)}P_{n_q}$ is fulfilled. By linearity, we find that either

$$B^{(k)}(C^+)_{n_p} \subset (C^+)_{n_q} = (C^+)_{n_p}$$

or

$$B^{(k)}(C^+)_{n_p} \subset -(C^+)_{n_q} = -(C^+)_{n_p}.$$

Therefore the operator $G = (B^{(k)})^2$ satisfies $G(C^+)_{n_p} \subset (C^+)_{n_p}$.

Consider the convex cone $K = K(P_{n_p})$ constructed from $(C^+)_{n_p}$. Again, we clearly see that $(B^{(k)})^2 K \subset K$. Furthermore, since

$$(B^{(k)})^2(x^1 \wedge \cdots \wedge x^k) \in \overset{\circ}{K}$$

for any nonzero $x^1 \wedge \cdots \wedge x^k \in (C^+)_{n_p}$, we deduce that

$$(B^{(k)})^2(K - \{0\}) \subset \overset{\circ}{K}.$$

That is, $(B^{(k)})^2$ is strongly positive with respect to K. We finish by noticing that

$$(B^{(k)})^2 = (A^{2(n_q - n_p)})^{(k)}$$

and thus the theorem holds with $n_0 = 2(n_q - n_p)$. $\square$

With this theorem, we get a sort of Perron–Frobenius theorem for this class of operators.

Theorem 2. *Let*

$$\mathrm{Sp}(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$$

be the spectrum of A with $|\lambda_i| \geqslant |\lambda_j|$ *for* $i < j$. *Then* $|\lambda_k| > |\lambda_{k+1}|$. *If we call H and* H^c *the generalized subspaces associated with* $\{\lambda_1, \ldots, \lambda_k\}$ *and* $\{\lambda_{k+1}, \ldots, \lambda_N\}$ *respectively, then*

$$H - \{0\} \subset \overset{\circ}{C} \quad \text{and} \quad H^c \cap C = \{0\}.$$

We shall use the next lemma.

Lemma 6. *Let L be a linear operator in* $\mathbb{R}^N$ *and let us assume that for certain* $n_0 \in \mathbb{N}$ *the power* L^{n_0} *has a leading positive eigenvalue* μ_0 *that is algebraically simple. Then L has a leading real eigenvalue* λ_0 *that is algebraically simple and* $\lambda_0^{n_0} = \mu_0$. *In addition the eigenspaces associated with* μ_0 *and* λ_0 *are equal.*

Proof. It is well known that if the spectrum of L is

$$\mathrm{Sp}(L) = \{\lambda_1, \ldots, \lambda_N\}$$

then the spectrum of L^{n_0} is just

$$\mathrm{Sp}(L^{n_0}) = \left\{\lambda_1^{n_0}, \ldots, \lambda_N^{n_0}\right\}.$$

Besides the generalized eigenspace associated with $\lambda_i^{n_0}$ is the direct sum of the generalized subspaces associated to the eigenvalues λ_j which satisfies $\lambda_j^{n_0} = \lambda_i^{n_0}$. From this the lemma straightforwardly follows. □

Proof of Theorem 2. From Theorem 1, we know that $G^{(k)}$ is strongly positive with respect to a convex cone $K = K(P)$, where $G = A^{n_0}$ for certain $n_0 \in \mathbb{N}$. The classical Perron–Frobenius theorem shows that $G^{(k)}$ has a leading positive eigenvalue that is algebraically simple. Moreover, it has an associated eigenvector that is positive. From Lemma 6, we deduce that $A^{(k)}$ has a leading eigenvalue that is algebraically simple and has a positive eigenvector.

Let us recall that the spectrum of $A^{(k)}$ consists of the products of k eigenvalues of A. Thus we straightforwardly find that the product $\lambda_1\lambda_2 \cdots \lambda_k$ is strictly dominant and simple. Hence necessarily $|\lambda_{k+1}| < |\lambda_k|$.

Let us now see the assertions concerning the eigenspaces H and H^c. First, we assume that A is diagonalizable (in the complex sense). In particular, there is a basis $\{v^1, \ldots, v^k\}$ of H such that either $Av^i = \lambda_i v^i$ if λ_i is a real eigenvalue or

$$Av^i = \alpha_i v^i + \beta_i v^{i+1}, \quad Av^{i+1} = -\beta v^i + \alpha v^{i+1}$$

provided that $\lambda_i = \alpha_i + i\beta_i$ and $\lambda_{i+1} = \alpha_i - i\beta_i$ are complex conjugate eigenvalues ($\beta_i \neq 0$). From this, we deduce that

$$Av^i \wedge Av^{i+1} = (\alpha^2 + \beta^2) v^i \wedge v^{i+1} = \lambda_i \lambda_{i+1} v^i \wedge v^{i+1}.$$

Hence

$$A^{(k)} v^1 \wedge \cdots \wedge v^k = Av^1 \wedge \cdots \wedge Av^k = \lambda_1 \cdots \lambda_k v^1 \wedge \cdots \wedge v^k.$$

Therefore $v^1 \wedge \cdots \wedge v^k$ is just an eigenvalue associated with the dominant eigenvalue $\lambda_1 \cdots \lambda_k$. Let us prove that $H = L[v^1, \ldots, v^k] - \{0\}$ is contained in $\overset{\circ}{C}$.

Let us take $w = x^1 \wedge \cdots \wedge x^k \in K - \{0\}$ such that $L[x^1, \ldots, x^k] \subset C$. It is well known that the sequence $w_n = \frac{G^n(w)}{\|G^n(w)\|}$ converges to a unitary eigenvector colinear to $v^1 \wedge \cdots \wedge v^k$. We can write $w_n = x_n^1 \wedge \cdots \wedge x_n^k$ and given that G is strongly positive we find that

$$L\left[x_n^1, \ldots, x_n^k\right] - \{0\} \subset \overset{\circ}{C} \quad \text{for all } n \in \mathbb{N}.$$

Again, since C is closed, we deduce that $L[v^1, \ldots, v^k] \subset C$. In fact, since $A(C - \{0\}) \subset \overset{\circ}{C}$, it follows that actually $L[v^1, \ldots, v^k] - \{0\} \subset \overset{\circ}{C}$.

The assertion concerning the eigenspace H^c follows from Lemma 4 and the same proof as above.

If A is not diagonalizable, it is well known that there exists a sequence $\{A_n\}$ of diagonalizable applications converging to A. It is easy to see that A_n is strongly positive with respect to C and inversible for large enough n. A straightforward continuity argument leads to the desired result. $\square$

Obviously, the cone $K(P)$ of Theorem 1 is equal to the cone induced by the invariant subspaces H and H^c of Theorem 2. We shall denote it by $K(A)$. Let us recall that by choosing $\{e^1, \dots, e^k\}$ a basis of H and $\{e^{k+1}, \dots, e^N\}$ a basis of H^c, the hyperplane P of Λ_k is generated by

$$B_k = \{e^{i_1} \wedge \cdots \wedge e^{i_k} : (i_1, \dots, i_k) \neq (1, \dots, k)\}.$$

Since $A(H^c) \subset H^c$, for each $w \in B_k$ the form $A^{(k)}w$ for $w \in B_k$ can never be equal to $e^1 \wedge \cdots \wedge e^k$. This means that P is invariant for $A^{(k)}$. Hence $A^{(k)}C^+(P)$ is equal either to $C^+(P)$ or $-C^+(P)$.

Corollary 1. *Let H and H^c be the invariant subspaces for A provided by Theorem 2. Then either $A^{(k)}$ or $(A^{(k)})^2$ is strongly positive with respect to the convex cone $K(A)$.*

Two results on continuous dependence of the cone K with respect to the operator A should me remarked upon. We denote by $\mathbb{S}L(\mathbb{R}^N)$ the space of linear inversible operator over $\mathbb{R}^N$.

Proposition 5. *There exists U neighborhood of A in $\mathbb{S}L(\mathbb{R}^N)$ such that $K(B) = K(A)$ for all $B \in U$.*

Proof. It is direct from Remark 3. $\square$

The second result establishes that the corresponding cone is preserved through special homotopies.

Definition 5. We say that two inversible operators B, C are strongly positively isotopic if there is a continuous function

$$M : [0, 1] \to \mathbb{S}L(N)$$

such that $M(0) = B$, $M(1) = C$ and $M(t)$ is strongly positive with respect to C for all $t \in [0, 1]$.

Proposition 6. *If B is strongly positively isotopic to A, then $K(B) = K(A)$.*

Proof. If we use $P(t)$ to refer to the hyperplane induced by $M(t)$, then $P(t)$ is continuous in t. Since $P(t) \cap C^{(k)} = \{0\}$ the proposition easily follows.

Remark 4. Definition 5 can be relaxed so that it allows $M(1)$ not to be strongly positive. For example, we can say that B is strongly positively isotopic to the identity if $M(1) = I$. In such a case, the leading product of the eigenvalues of B immediately becomes positive.

5. A particular case

In this section, we exemplify our work with a well-known cone of rank k.

Given $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ having no zero component, we define $\mathrm{sign}(x)$ as the number of sign changes in the sequence $x_1, \dots, x_N$. The set

$$C_k = \overline{\{x : \mathrm{sign}(x) \leqslant k - 1\}}$$

is a cone of rank k for $k = 2, \dots, N - 1$. Moreover, it is k-solid and complemented (see [1,7]).

Our aim in this section is to establish the following result:

Theorem 3. *The unique convex cone of Λ_k associated to C_k is K^+ the cone of forms having non-negative coordinates with respect to the usual basis of Λ_k. In addition, C_k is full and k-connected.*

For the sake of completeness, we shall give a proof through a series of lemmas, though many of the arguments used are similar to those appearing in [1,7]. For instance, the first one is borrowed from Krasnoselskij et al. [7].

Lemma 7. *If* $x^1, \ldots, x^k \in \mathbb{R}^N$ *satisfy that* $x^1 \wedge \cdots \wedge x^k \in \overset{\circ}{K}{}^+$, *then* $L[x^1, \ldots, x^k] - \{0\} \subset \overset{\circ}{C}_k$ *.*

Proof. Let us suppose that the lemma is not true. This means that there is

$$y = (y_1, \ldots, y_N) \in L[x^1, \ldots, x^k] - \{0\}$$

such that

$$(-1)^j y_{i_j} \geqslant 0 \quad \text{for } 1 \leqslant i_1 < i_2 < \cdots < i_{k+1} \leqslant N.$$

In addition since $y \neq 0$ at least one of the products $(-1)^j y_{i_j}$ is strictly positive. Consider the obvious equality

$$0 = \operatorname{Det}\begin{pmatrix} y_{i_1} & x^1_{i_1} & \ldots & x^k_{i_1} \\ y_{i_2} & x^1_{i_2} & \ldots & x^k_{i_2} \\ \ldots & \ldots & \ldots & \\ y_{i_{k+1}} & x^1_{i_{k+1}} & \ldots & x^k_{i_{k+1}} \end{pmatrix}. \tag{8}$$

Expanding the determinant by the first column, we have by hypothesis a sum of non-negative numbers where at least one is strictly positive. This makes (8) impossible. □

Lemma 8. *If* $L[x^1, \ldots, x^k] - \{0\} \subset \overset{\circ}{C}_k$ *then* $x^1 \wedge \cdots \wedge x^k \in \overset{\circ}{K}{}^+ \cup (-\overset{\circ}{K}{}^+)$.

Proof. First of all if $x^1 \wedge \cdots \wedge x^k$ has a zero component, there exists $y \in L[x^1, \ldots, x^k] - \{0\}$ having k zero components. From this, it is immediate that close to y there are vectors having just k sign changes; this contradicts that $y \in C_k$. Therefore $x^1 \wedge \cdots \wedge x^k$ must have no zero component. In the sequel, we reason by reduction to the absurd and assume that $x^1 \wedge \cdots \wedge x^k$ have components of both signs.

For the rest of the proof, we use a recurrent argument in $k = 2, \ldots, N-1$. Let us first consider the case $k = 2$. We then take $x^i = \left(x^i_1, \ldots, x^i_N\right)$, $i = 1, 2$ such that $x^1 \wedge x^2$ has a positive component and a negative component. We may assume that x^1_1 and x^2_1 are not simultaneously zero. If this is not the case, we can eliminate the initial zero components since the following reasoning still holds.

By taking a suitable linear combination of x^1 and x^2, we can assume that $x^1_1 = 1$ and $x^2_1 = 0$. We can also assume that

$$\operatorname{Det}\begin{pmatrix} 1 & 0 \\ x^1_j & x^2_j \end{pmatrix} = x^2_j > 0$$

for certain j. Since $x^2 \in \overset{\circ}{C}_2$ we deduce that

$$x^2_i > 0 \quad \text{for all } i = 2, \ldots, N. \tag{9}$$

By hypothesis, there are subindices $1 < p < q$ such that

$$d = \operatorname{Det}\begin{pmatrix} x^1_p & x^2_p \\ x^1_q & x^2_q \end{pmatrix} < 0. \tag{10}$$

Let us consider $\alpha^1,\ \alpha^2$ the solutions of the system

$$\begin{cases} \alpha^1 x^1_p + \alpha^2 x^2_p = -1, \\ \alpha^1 x^1_q + \alpha^2 x^2_q = 1. \end{cases}$$

We find that

$$\alpha^1 = \frac{\operatorname{Det}\begin{pmatrix} -1 & x_p^2 \\ 1 & x_q^2 \end{pmatrix}}{d} = \frac{-x_p^2 - x_q^2}{d}.$$

From (9) and (10) we deduce that $\alpha^1 > 0$. Therefore the vector

$$v = \alpha^1 x^1 + \alpha^2 x^2 \in L[x^1, x^2]$$

has at least two sign changes (at the coordinates 1st, pth and qth). This means that $L[x^1, x^2]$ is not contained in C_2.

Let us assume that we have proved the result for $\bar{k} = 2, \ldots, k-1$ and let us prove it for $k > 2$.

As before, we can assume that x_1^j is nonzero for some $j = 1, \ldots, k$. Furthermore, by performing some linear combinations, we can assume that $x_1^1 = 1$ and $x_1^i = 0$ for $i = 2, \ldots, k$. These equalities imply that the linear combinations of the vectors $x^2, \ldots, x^k$ can have at most $k-1$ sign changes. Again the fact that $L[x^1, \ldots, x^k] - \{0\} \subset \overset{\circ}{C}_k$ also implies that the minors of the form

$$\operatorname{Det}\begin{pmatrix} x_{i_1}^2 & \ldots & x_{i_1}^k \\ \ldots & \ldots & \ldots \\ x_{i_{k-1}}^2 & \ldots & x_{i_{k-1}}^k \end{pmatrix}$$

are all nonzero.

From the recurrent assumption, we know that the preceding minors all have the same sign. After a permutation of the vectors, we can assume that these minors actually satisfy

$$\operatorname{Det}\begin{pmatrix} x_{i_1}^2 & \ldots & x_{i_1}^k \\ \ldots & \ldots & \ldots \\ x_{i_{k-1}}^2 & \ldots & x_{i_{k-1}}^k \end{pmatrix} > 0. \tag{11}$$

By hypothesis, there is a minor

$$d = \operatorname{Det}\begin{pmatrix} x_{i_1}^1 & \ldots & x_{i_1}^k \\ \ldots & \ldots & \ldots \\ x_{i_k}^1 & \ldots & x_{i_k}^k \end{pmatrix} < 0. \tag{12}$$

This minor cannot include the first file $(1, 0, \ldots, 0)$, since these are all positive. Let us consider $\alpha^1, \ldots, \alpha^k$ the solutions of the system

$$\begin{cases} \alpha^1 x_{i_1}^1 + \ldots + \alpha^k x_{i_1}^k = -1, \\ \qquad\qquad\qquad \ldots = 1, \\ \alpha^1 x_{i_k}^1 + \ldots + \alpha^k x_{i_k}^k = (-1)^k. \end{cases}$$

We know that

$$\alpha^1 = \frac{\operatorname{Det}\begin{pmatrix} -1 & \ldots & x_{i_1}^k \\ 1 & \ldots & x_{i_2}^k \\ \ldots & \ldots & \ldots \\ (-1)^k & \ldots & x_{i_k}^k \end{pmatrix}}{d} > 0.$$

By expanding the numerator with respect to the first column and applying (11) and (12) we get that $\alpha^1 > 0$.

Hence, the vector $v = \alpha^1 x^1 + \cdots + \alpha^k x^k$ again has k sign changes at least and then we reach the desired contradiction. $\square$

We shall now see a connectivity property linked to Lemma 3.

Lemma 9. *The set of decomposable forms in $K^+ - \{0\}$ is connected.*

Proof. Let us take any form $x^1 \wedge \cdots \wedge x^k \in K^+ - \{0\}$ and $i_1 < \cdots < i_k$ such that

$$m(i_1, \ldots, i_k) = \operatorname{Det}\begin{pmatrix} x^1_{i_1} & \ldots & x^k_{i_1} \\ \ldots & \ldots & \ldots \\ x^1_{i_k} & \ldots & x^k_{i_k} \end{pmatrix} > 0.$$

Given $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \geqslant 0$ for $i = 1, \ldots, N$, let us consider the form

$$\lambda(\alpha) = x^1(\alpha) \wedge \cdots \wedge x^k(\alpha),$$

where $x^i(\alpha) = \left(\alpha_1 x^i_1, \ldots, \alpha_N x^i_N\right)$. It is immediate that $\lambda(\alpha) \in K^+$ for every α. Let us consider the set

$$\mathbb{S}(x^1 \wedge \cdots \wedge x^k) = \left\{\lambda(\alpha) : \begin{cases} \alpha_{i_j} = 1 & \text{for } j = 1, \ldots, k \\ 0 \leqslant \alpha_m \leqslant 1 & \text{otherwise} \end{cases}\right\}.$$

It turns out that $\mathbb{S}(x^1 \wedge \cdots \wedge x^k)$ is a connected set containing the forms

$$x^1 \wedge \cdots \wedge x^k \quad \text{when } \alpha = (1, 1, \ldots, 1)$$

and

$$(0, \ldots, 0, m(i_1, \ldots, i_k), 0, \ldots, 0) \quad \text{when } \alpha_m = 0 \text{ for } m \neq i_j,\ j = 1, \ldots, k.$$

In particular, each decomposable form in K^+ can be connected with any form in $\overset{\circ}{K}{}^+$. This proves our assertion. $\square$

We now wish to prove that C_k is full. To do so, let us consider the parametric family of tridiagonal matrices of order N

$$A(r) = \begin{pmatrix} 1 & r & 0 & 0 & \ldots & 0 \\ r & 1 & r & 0 & \ldots & 0 \\ 0 & r & 1 & r & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & 0 & r & 1 \end{pmatrix}, \quad r > 0.$$

For all small enough r the matrix $A(r)$ is oscillating. Hence, for each k, there is a positive integer p such that $A(r)^p$ is strongly positive with respect to C_k (see [7]).

Lemma 10. *The operator $(A(r)^p)^{(k)}$ is strongly positive with respect to K^+.*

Proof. Let $B = \{e^1, \ldots, e^N\}$ of $\mathbb{R}^N$ be the standard basis of $\mathbb{R}^N$. For a multiindex $(i) = (i_1, \ldots, i_k)$ we know that

$$H(i) = L[e^{i_1}, \ldots, e^{i_k}] \subset C_k.$$

The strong positiveness of $A(r)^p$ yields to

$$A(r)^p H(i) - \{0\} \subset \overset{\circ}{C}_k .$$

In particular

$$\lambda(i) = (A(r)^p)^{(k)} e^{i_1} \wedge \cdots \wedge e^{i_k} \subset \overset{\circ}{K}{}^+ \cup (-\overset{\circ}{K}{}^+).$$

Now, we notice that $A(r)$ (and thus $(A(r)^p)^{(k)}$) tends to the identity when r tends to 0. Therefore

$$\lambda(i) \subset \overset{\circ}{K}{}^+$$

at least for small enough r. Since all the extreme rays of K^+ are of the form $e^{i_1} \wedge \cdots \wedge e^{i_k}$ we immediately obtain that $(A(r)^p)^{(k)}K - \{0\} \subset \overset{\circ}{K}{}^+$. $\square$

Corollary 2. *If* $x^1 \wedge \cdots \wedge x^k \in K^+$ *then* $L[x^1, \ldots, x^k] \subset C_k$.

Proof. Due to Lemma 7 we consider only the case that $x^1 \wedge \cdots \wedge x^k$ belongs to the boundary of K^+. Then

$$(A(r)^p)^{(k)}x^1 \wedge \cdots \wedge x^k \in \overset{\circ}{K}{}^+$$

from the preceding lemma. Lemma 8 shows that

$$A(r)^p L[x^1, \ldots, x^k] - \{0\} \subset \overset{\circ}{C}_k .$$

Since $A(r)$ tends to the identity when r tends to zero and C_k is closed, we deduce that $L[x^1, \ldots, x^k]$ is contained in C_k. $\square$

Proof of Theorem 3. Lemma 3 implies that any hyperplane P yielding to the convex cone $K(P)$ cannot intersect $K^+ \cup (-K^+)$. Therefore

$$C_+^{(k)}(P) = C^{(k)}(P) \cap K^+$$

because of Lemmas 7, 8 and Corollary 2. The rest of the theorem is straightforward from the stated results. $\square$

6. An application to autonomous differential equations

The basis of this paper is found in [10], where a new class of smooth autonomous systems

$$\dot{x} = F(x), \quad x \in \mathbb{R}^N \tag{13}$$

is introduced, called C-cooperative systems. Here, C stands for a complemented cone of rank 2 that is 2-solid. The main property fulfilled by such a systems is the Poincaré–Bendixson property enjoyed by some of its orbits.

In this section, we outline how the results described above can be employed in the study of the stability of periodic orbits.

Given $x(t)$ a nonconstant T-periodic solution of (13), its variational equation is the linear system

$$\dot{u} = F'(x(t))u. \tag{14}$$

Let $U(t)$ be the matrix solution of (14) satisfying $U(0) = I$. The operator $U(T)$ is called the monodromy operator associated with $x(t)$. The spectrum of $U(T)$ always contains the point $\mu_1 = 1$. If we denote the remaining eigenvalues by $\mu_2, \ldots, \mu_N$, it holds that $x(t)$ is an (orbitally) attracting solution if $|\mu_i| < 1$.

Let us assume then that system (13) is C-cooperative. This means that in particular $U(t)$ is strongly positive with respect to C for all $t > 0$ (see [10]). In addition, $U(t)$ forms a family of strongly positively isotopic to the identity. By Proposition 6 the convex cone K induced by $U(t)$ is independent of $t > 0$. The operator $U(t)^{(2)}$ is strongly positive with respect to K for $t > 0$ and its dominant eigenvalue is $\mu_1\mu_2 = \mu_2$. Putting all this together, we can state the following proposition.

Proposition 7. *The nonconstant T-periodic solution $x(t)$ is orbitally attracting provided that $U(T)^{(2)}$ has a spectral radius lesser than* 1. *In addition, the eigenvalue μ_2 is positive.*

Thus we have reduced the problem of the stability of $x(t)$ to the classical problem of estimating the spectral radius of a positive operator. In concrete examples $U(T)^{(2)}$ is not explicitly known but it satisfies a linear system

$$\dot{y} = F'(x(t))^{[2]}y, \tag{15}$$

where $A^{[k]}$ stands for the k-additive compound of the matrix A (see [3,11]). Proceeding as in [8] or [9] one can obtain the desired estimates.

References

[1] T. Ando, Totally positive matrices, Linear Algebra Appl. 90 (1987) 165–219.
[2] A. Berman, R.J. Plemmons, Nonnegative Matrices in Mathematical Sciences, SIAM, Philadelphia, 1994.
[3] V.A. Boichenko, G.A. Leonov, V. Reitmann, Dimension Theory for Ordinary Differential Equations, Teubnes Texts in Mathematics, vol. 141, Stuttgart, 2005.
[4] G. Fusco, M.W. Oliva, A Perron theorem for the existence of invariant subspaces, Ann. Mat. Pura Appl. 160 (1991) 63–76.
[5] F.R. Gantmacher, M.G. Krein, Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme, Akademie-Verlar, Berlin, 1960.
[6] M.W. Hirsch, H.L. Smith, Monotone Dynamical Systems, Handbook of Differential Equations: Ordinary Differential Equations, vol. II, Elsevier B.V., Amsterdam, 2005, pp. 239–357.
[7] M.A. Krasnoselskij, J.A. Lifshits, A.V. Sobolev, Positive Linear Systems, Heldermann Verlag, Berlin, 1989.
[8] L.A. Sanchez, Dynamics of the modified Michaelis–Menten system, J. Math. Anal. Appl. 317 (2006) 71–79.
[9] L.A. Sanchez, Global asymptotic stability of the Goodwin system with repression, Nonlinear Anal. Real World Appl. 10 (2009) 2151–2156.
[10] L.A. Sanchez, Cones of rank 2 and the Poincaré–Bendixson property for a new class of monotone systems, J. Differential Equations 216 (2009) 1170–1190.
[11] B. Schwarz, Totally positive differential systems, Pacific J. Math. 32 (1970) 203–229.
[12] H.L. Smith, Monotone Dynamical Systems, American Mathematical Society, Providence, 1995.
[13] R.H. Wasserman, Tensors & Manifolds, Oxford University Press, Oxford, 1992.